Theory of chaos in a space-time perspective

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1988

SERIE RESEARCH MEMORANDA

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Researchmemorandum 1988-29 augustus 1988

VRIJE UNIVERSITEIT
FACULTEIT DER ECONOMISCHE WETENSCHAPPEN
EN ECONOMETRIE
AMSTERDAM

THEORY OF CHAOS IN A SPACE-TIME PERSPECTIVE

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ABSTRACT

In recent years chaos theory has received a great deal of attention among social scientists. Chaos theory originated from turbulent-type of motions in physical systems, but recently its relevance has been explored successfully in economic and social sciences.

After a brief overview of recent developments related to chaos theory in economic systems, our paper will present a chaotic model for urban forms based on socio-economic push-pull activities. Its stability properties will also be investigated. In particular, we will study a chaotic model for urban decline emerging from so-called Lorenz equations. A new element will be the combination of optimal control theory and chaos theory.

1. What is Chaos?

In recent years, a great many books and articles on the principle of chaos have been published. The discovery of 'chaos' is sometimes regarded as a scientific novelty comparable to Newton's laws of motion in physics. Chaotic behaviour takes place when relatively small stimuli in a complex dynamic system cause unpredictable - or at least seemingly unexpected - responses, even if the system concerned were originally in an equilibrium state.

The interesting feature of a chaotic system is that its dynamics cannot be attributed to stochastic fluctuations (in certain key parameters, for instance), but that its evolution obeys (usually simple) deterministic rules which can easily generate wild and seemingly random fluctuations (see e.g. Crutchfield et al., 1986). At present the notion of 'chaos' refers to deterministic but hardly predictable system's behaviour.

The analysis of chaotic systems has become a fascinating activity of scientists in various disciplines, and the visual presentation of chaotic systems by means of modern computer-graphics has led to very imaginative and speculative inferences (see also Peitgen and Richter, 1986). However, various scientists appear to use different definitions and formal representations of chaotic behaviour, and - given this lack of unambiguity - it may be worth describing in more detail the origins and the historical evolution of the concept of 'chaos'.

The behaviour of deterministic non-periodic flows has been investigated for the first time by Lorenz (1963), who studied the instability of such flows for forced dissipative hydrodynamical systems and was able to derive the numerical solutions for these systems by means of the convection equations of Saltzman. In particular, he found that the projections of the solution trajectories of such dynamic systems followed two spirals -around two steady states - at different surfaces, so that << it is possible for the trajectory to pass back and forth from one spiral to the other without intersecting itself >> (Lorenz, 1963, p. 138).

In subsequent discussions on the concept of chaos, the related term 'strange attractors' was introduced, first by Ruelle and Takens (1971), in order to indicate << an exponential separation of orbits (as time goes on) of points which initially are very close to each other >> (see Eckmann and Ruelle, 1985, p. 619). In fact, the Lorenz model may be regarded as the first example in the scientific literature having strange attractors.

Later on, Li and Yorke (1975) christened 'chaotic' a system with strange attractors or a dynamic situation exhibiting aperiodic - though bounded - trajectories. The term 'chaos' was only used before by the physicist Boltzmann, be it in a different context.

In general, a chaotic system refers to dynamic phenomena marked by the occurrence of unknown random dynamics - and hence by unpredictability - in completely deterministic systems, mainly in the context of dissipative systems. It is noteworthy here that often - instead of the term 'chaos' - alternative expressions with the same meaning are used by various authors, such as 'dynamical stochasticity', 'self-generated noise', or 'intrinsic stochasticity' (see Hao, 1984).

After these introductory remarks, we now need a more strict definition of chaos. Ott (1981) and Pacini (1986) define a system as chaotic, if there exists an uncountable, invariant set A of initial conditions such that all trajectories starting in A meet the following requirements:

- they never repeat themselves (aperiodic elements of A);
- they neither attract nor are attracted by other trajectories;
- they show a sensitive dependence on initial conditions.

Moreover, together with chaotic trajectories, periodic points of every order coexist, although most of them are not acting as an attractor. In practice, small uncertainties or small perturbations are amplified, so that - even if the behaviour is predictable in the short term - it is unpredictable in the long term and it may lead to very different trajectories. As a consequence it is usually impossible to make accurate predictions for unstable behaviour for other than very short time horizons. Thus the resulting aperiodic and interlaced cycles are able to produce very complicated forms.

The discovery of 'chaos' seems to have created a new paradigm in scientific modelling. Firstly, the process of verifying theories on dynamic systems behaviour through conventional predictions becomes more problematic in case of chaotic systems. And secondly, the concept of chaos demonstrates that a system can have a complex global behaviour at large which in general cannot be deduced from knowledge of its constituent parts. Thus << chaos provides a mechanism that allows for free will within a world governed by deterministic laws >> (see Crutchfield et al., 1986, p. 57).

As mentioned above, the theory of chaos has attracted a great deal of scientific attention, not only in the form of speculative articles in popular journals, but also in the form of more substantial scientific contributions, notably in two volumes encompassing various basic papers on this issue, viz. Cvitanovic (1984) and Hao (1984). In the next section, a concise overview of some of the most relevant scientific contributions will be presented.

2. Strange Attractors: A Brief Overview

For a better understanding of chaotic behaviour, which represents nowadays the main body of new theoretical ideas concerning non-linear dynamics, it is interesting to discuss briefly the most interesting contributions on strange attractors presented thus far.

2.1. May

May (1976) discussed a very simple - but later on extremely popular - logistic equation applied to a biological population X:

$$X_{t+1} = a\, X_t(1-X_t) \qquad 0<X<1 \qquad (2.1.)$$

where the parameter a is reflecting the maximum per capita rate of increase.

It can be demonstrated that for particular values of a ($1<a<4$) chaotic patterns will emerge. Eq. (2.1.) also exhibits fixed points as well as bifurcations of fixed points (see also Figure 1). In particular, according to Li and Yorke (1975), there is a value of a (a = 3,8284..) for which a fixed point of period three exists and a subset A exists as defined in section 1.

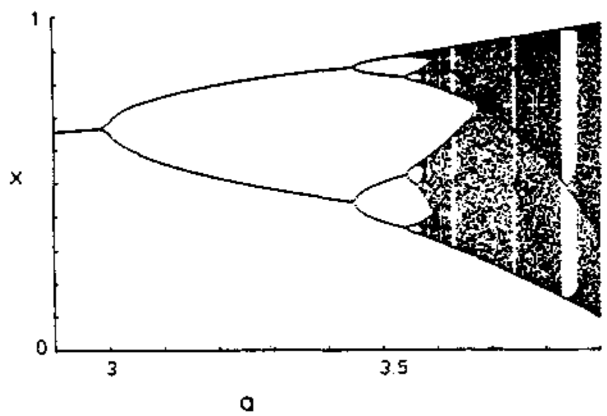

Figure 1. Bifurcation diagrams of a May logistic map ($2.9<a<3.9$).
Source: Holden (1986, p. 46)

2.2. Hénon

Hénon (1976) has found a strange attractor in a two-dimensional quadratic mapping. In his contribution he clearly identifies a strange attractor by regarding it as a volume of a flow (in three-dimensional space) shrinking exponentially over time.

Moreover, there exists a bounded region towards which every trajectory tends to move (attractor set). The attractor is a point or a closed curve, or it can have a more complex structure. This last case defines the strange attractor. << Inside the attractor, trajectories wander in an apparently erratic manner. Moreover, they are highly sensitive to initial conditions >> (Hénon, 1976, p. 69).

Hénon starts from the following system of difference equations, describing a dynamic physical, chemical or biological system:

$$\left.\begin{aligned} x(t+1) &= y(t)+1-ax(t)^2 \\ y(t+1) &= bx(t) \end{aligned}\right\} \qquad (2.2.)$$

Given x(0), y(0) and by selecting particular values of a and b (a=1.4; b=0.3) he finds an attractor consisting of a number of more or less parallel "curves" (see Figure 2).

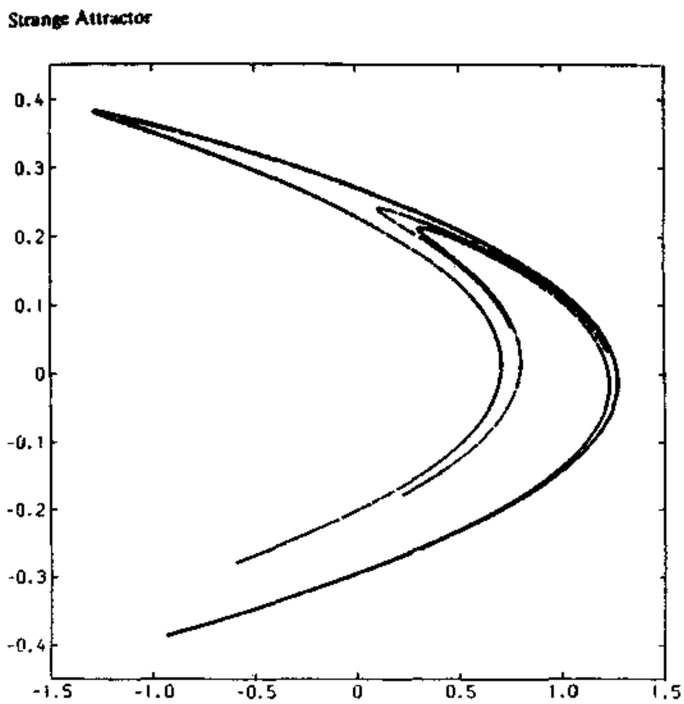

Figure 2. The Hénon attractor.
Source: Hénon (1976, p. 73)

The Hénon attractor is a strange attractor because it is chaotic (i.e., with sensitive dependence on initial conditions) and because it has fractal characters (see Eckmann and Ruelle, 1985). Fractal is a term coined by Mandelbrot (1977). Roughly speaking a fractal set is a set having the property of being invariant at different scales (self-similarity and irregularity property) and consequently having not an integral dimension (for an application of fractal geometry to urban structure see Batty and Longley, 1986). Therefore, the notion of fractals only refers to the geometry of attractors (see also Mandelbrot, 1977, and Peitgen and Richter, 1986).

Many authors define strange attractors only by referring to the dynamics of the attractors and not just to its geometry. Therefore, strange attractors need not have a fractal structure and attractors with a fractal structure need not be chaotic (see e.g., Holden and Muhamad, 1986).

It should also be noted that the minimum dimensionality of a continuous time dynamical system which is able to generate chaotic time paths, is equal to three (i.e., $n=3$). This result follows from the Poincaré-Bendixon theorem (see Lichtenberg and Lieberman, 1983 and Lorenz, 1986). The chaotic motion is therefore associated with the existence of homoclinic and heteroclinic orbits (see Sparrow, 1982 and Weiss, 1987), when a limit cycle or a torus collides with a non-stable singular point.

It is interesting to illustrate next some further examples of strange attractors arising from continuous differential systems, rather than from discrete systems (like the May model). This will be done in the next two subsections.

2.3. Gilpin

An interesting system studied in an ecological context is the well-known Lotka-Volterra system of the following form:

$$[1|X_i(t)]\ \dot{X}_i(t) = r_i + \sum_j a_{ij} X_j(t) \qquad (2.3.)$$

Gilpin (1979) demonstrates that equation (2.3.), which models the dynamics of a single predator and two prey species, can give rise to chaotic trajectories (see Schaffer and Kot, 1986) (see also Figure 3).

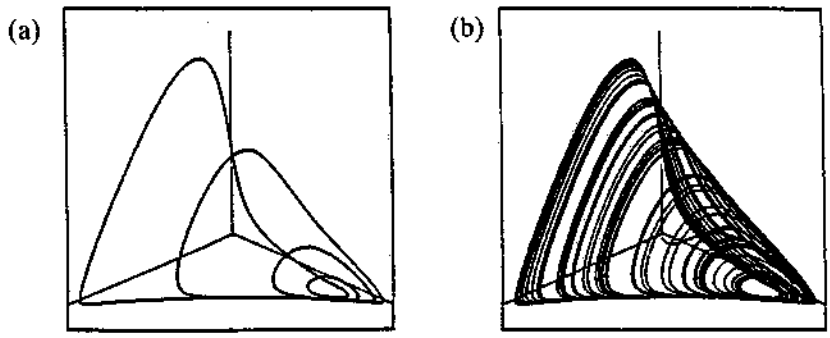

Figure 3. The chaotic region for Gilpin's equations.
Source: Holden (1986, p. 164)

In particular the chaotic region for Gilpin's equations contain periodic orbits which can be identified with the logistic ones.

2.4. <u>Lorenz</u>
As we pointed out in section 1, the Lorenz attractor is the first (and therefore the best-known) example of a strange attractor. Lorenz (1963) considers the following differential system describing a horizontal fluid layer heated from below and cooled from above:

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -xz+rx-y \\ \dot{z} &= xy-bz \end{aligned}\right\} \qquad (2.4)$$

where in his original hydrodynamical system x represents the convective motion, y the horizontal temperature variation and z the vertical temperature variation. The parameters σ, r and b are the Prandtl number, the Rayleigh number and the size of the region, respectively.

For particular values of the parameters, viz. $\sigma = 10$, $r = 28$ and $b = 8/3$, Lorenz finds complicated attractors with trajectories spiralling around, and jumping between, two loops (see Figure 4).

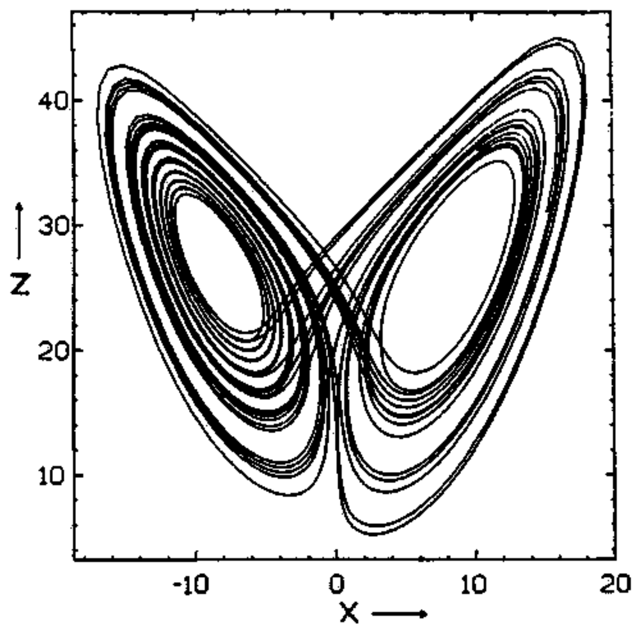

Figure 4. The Lorenz attractor for $r = 28$, $b = 8/3$, $\sigma = 10$. The trajectory is projected on the x, z plane.
Source: Holden (1986, p. 126)

2.5. <u>Rössler</u>

Rössler (1976) studied a simple three-dimensional system which models the flows around one of the loops of the Lorenz attractor:

$$\left.\begin{aligned}\dot{x} &= -(y+z)\\ \dot{y} &= x+ay\\ \dot{z} &= b+z(x-c)\end{aligned}\right\} \qquad (2.5.)$$

In the classical form, the parameters a and b are treated as constants (a=b=0.2), while the parameter c is treated as a bifurcation parameter. Chaos develops at the accumulation point of the period-doubling sequence from a simple, period-one oscillation, just above c=4.2. However, also for other values of the parameters chaos may appear, leading to slightly different forms from the previous one (see Holden and Muhamad, 1986) (see Figure 5). Rössler also developed from (2.5.) a four-dimensional system exhibiting a strange attractor (hyperchaos).

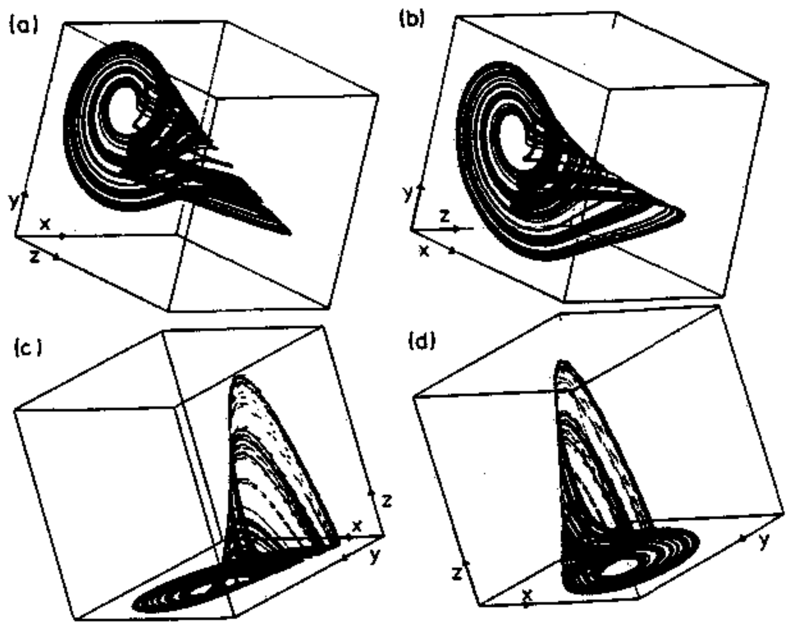

Figure 5. The Rössler attractor. Three-dimensional views with $a = 0.343$, $b = 1.82$ and $c = 9.75$.
Source: Holden (1986, p. 24)

It should be noted that the last two systems (2.4.) and (2.5.) contain a cross-term, in which the rate of change of one variable is related to a term that is the product of the other two variables (the so-called synergetic effect). Finally, it is worth mentioning that for differential systems of order 3 a 'strange attractor' is an object whose dimension is an intermediate between a surface and a volume, i.e., a surface with an infinite number of sheets and hence with fractal dimensions (see Hénon and Pomeau, 1976)

2.6. Concluding remarks

Non-linear dynamics has heralded many new directions in the analysis of dynamic systems governed by relatively simple rules. Turbulence has become a key feature of evolutionary research, not only in the natural but also in the social sciences. In this context, causality analysis seems to need a reorientation compared to the past, when it was assumed usually that small changes in initial conditions or parameters would only have small effects. Nowadays, the awareness is growing that linear cause-effect relationships - even in the case of incremental changes - are not always plausible, so that the specification of cause - effect models has to allow unexpected movements in a longer time horizon. Consequently, also equilibrium and stability analysis has to be re-interpreted from this perspective.

3. Regional Economic Applications

3.1. Introduction

It is evident from the previous examples that much thorough research work still needs to be done in both a theoretical and empirical respect. An important analytical problem inherent in chaos theory is the choice between a discrete (i.e., difference equation form) and a continuous (i.e., differential equation form) systems representation. Computationally, discrete dynamic systems have a richer spectrum of behaviour than the corresponding continuous systems (see, e.g., Pacini, 1986), so that the results can sometimes be completely different for these two specifications. For instance, the May model will normally exhibit only chaotic behaviour in case of a difference equation form.

It is noteworthy that in economics some dynamic systems which are easy to handle from an empirical viewpoint, notably one-dimensional discrete systems, have been applied largely in growth models or business cycle models (see, among others, Benhabib and Day, 1980, 1982; Day, 1982; Goodwin et al, 1984; Guckenheimer et al., 1977; Grandmont, 1984 and Stutzer, 1980). However, also some criticism on these types of models has been put forward. Firstly, they do not always describe adequately economic fluctuations, which have both a business and a structural component. Secondly, models generating chaotic behaviour include so far only a very limited number of equations, whereas in reality economic systems are usually much more complicated. And finally, in various cases the economic justification underlying the specification of these chaotic growth models is not always clear (see, e.g., Cugno and Montrucchio, 1984 and Pacini, 1986). In any case, it should be clear that in economics chaos or turbulence is not a necessity, but only a possibility, so that it is at least a valid research endeavour to specify models allowing for chaos.

In this context the interpretation given by Lorenz (1986) is worth mentioning. The author investigates a multisector (continuous) model of business fluctuations (as a special case of the cycle model of Kaldor); he shows that even though it is not conclusive that chaos always occurs in this type of model, there is at least <u>a chance</u> to face chaotic motions which consist, in this case, of coupled oscillators.

After these introductory considerations which show the need for a clear analysis of chaotic dynamics, especially in economic theory, we will now discuss a selection of applications devoted to one specific research area, viz. regional economics. This topic has drawn quite some attention in the recent past (see e.g., Domanski and Wierzbicki, 1983, and Lung, 1988). In our paper we will in particular show some interesting examples displaying chaotic behaviour in spatial systems, which include also some interesting theoretical aspects. Clearly, this overview is by no means meant to be exhaustive; it is rather illustrative and indicative.

3.2. <u>White</u>

White (1985) has looked for the conditions under which chaotic behaviour arises in an industrial system. In particular he models the growth (or decline) of each sector in each centre by using difference equations of the following type:

$$X_{ij,t+1} = X_{ij,t} + r_j \; (P_{ij,t}), \qquad (3.1.)$$

where $X_{ij,t}$ represents the size of sector j in centre i at time t, r_j the intrinsic growth rate of the sector, and $P_{ij,t}$ the profit generated (which is depending on the aggregate amount produced in the sector concerned by all centres).

It is evident that (3.1.) belongs to the family of Verhulst equations (see Peitgen and Richter, 1986) discussed also by May (1976) and Yorke and Yorke (1981) in the framework of chaotic behaviour.

The simulation results show that the value of r for which chaotic behaviour appears is inversely related to the number of the centers. Furthermore the author stresses that the onset of chaotic behaviour is not so clear as pointed out by previous authors. In his interesting contribution he investigates in particular different degrees of chaos for the one-equation models.

3.3. Dendrinos

Dendrinos has explored chaotic dynamics (mostly in socio-spatial systems) from both a theoretical and an empirical perspective. In a first article (Dendrinos, 1984), he uses a May-type of differential equation for modelling urban macro dynamics. More specifically he adopts the following form:

$$y(n+1) = A\, y(n)[B-y(n)] \qquad (3.2.)$$

where $y(n) \leq B$ represents the population and A, B > 0 are relevant parameters. The author shows that formulation (3.2.) satisfactorily replicates urban aggregate dynamics in the U.S. for the period 1890-1980. He observes in particular that the size of urban areas always affects (inversely) the amplitude or the number of the oscillations required to reach a steady state.

Furthermore, he demonstrates that the A, B values associated with U.S. cities are close to the turbulent regime in the (A,B) space, but temporarily remote from it. In a second contribution (Dendrinos, 1986) the author tries to overcome the problem of the choice between discrete and continuous dynamics by connecting spatial flows to continuous fluid convection dynamics on the basis of earlier research undertaken by Lorenz (1963) and Sparrow (1982). He then applies a Lorenz system to regional employment by adopting seven parameters which produce less efficiency than the Lorenz model. The results of this model show that the trajectories converge towards periodic orbits, although such orbits may not be well defined.

The conclusion seems that unexpected behaviour may exist depending on fluctuations in the model parameters induced from exogenous changes.

3.4. Dendrinos and Sonis

Dendrinos and Sonis (1988) recently investigated socio-spatial dynamics on the basis of a one-dimensional discrete map. The authors studied discrete regional relative population dynamics by following the line of research described in subsection 3.3. (Dendrinos, 1984). Their analysis shows the importance of the level of disaggregation used in the analysis of dynamic instability.

In another article (Dendrinos and Sonis, 1987) the authors explore the onset of turbulence in discrete relative multiple spatial dynamics by demonstrating local and partial turbulence. Furthermore they also show (see Sonis and Dendrinos, 1987) that the well-known Feigenbaum sequence does not hold over the bifurcation parameter sequence for period-doubling cycles.

3.5. Nijkamp

Nijkamp (1987) has developed a simple model for analyzing endogenous long-term spatial fluctuations. On the basis of a dynamic production function he ends up with the following relationship:

$$\Delta y_t = \tilde{y}_t(y^{max} - ky_{t-1})y_{t-1}/y^{max} \qquad (3.3.)$$

where y_t represents the regional share in the national production and $\tilde{y}_t$ is the rate of change in the original quasi-production function, incorporating infrastructure capital and R&D capital.

It is evident that (3.3.) is essentially a May-type model, so that (3.3.) is able to generate a wide variety of dynamic growth patterns, although in principle the behaviour of such a model is determined by the initial conditions of the system and by its growth rate.

In a subsequent paper (Nijkamp et al, 1988), the authors have extended the previous model toward a Harrod type of growth model by incorporating also investment and savings behaviour. Next, R&D investments are endogenized, by assuming that the growth path of income, consumption and investment is co-determined by R&D investments. By imposing next the condition of a declining marginal efficiency of R&D expenditures and finally even of a saturation level, one faces the possibility of diseconomies of scale. The (maximum) saturation level plays the same role as y^{max} in (3.3.). By means of various simulation experiments in both a single region and a multi-region system the authors were able to analyze the dynamic behaviour of a dynamic spatial economic system.

4. A Simple Model of Chaos for Urban Decline

In the present section a simple model will be developed which is able to generate - under certain conditions on parameters - chaotic behaviour. The model is assumed to reproduce the potential evolutionary pattern of a declining area. Empirical evidence on urban decline of many cities can be found in a great number of recent studies on urban evolution. Three key variables are assumed to play a basic role in our case, viz. city size (measured in terms of number of inhabitants), employment potential (measured in terms of employment rate, i.e. the share of working population in total population), and urban attractiveness (measured as the inmigration rate, i.e. the share of immigrants vis-à-vis total population). The latter variable may be negative in case of urban repulsion effects. Instead of employment potential, one might also use supply minus demand for jobs.

In case of a declining area, the growth rate of population is negative, although this may be compensated by a rise in employment from outside the urban system. Consequently, we may assume the following simple relationship:

$$\dot{x} = \sigma_1 y - \sigma_2 x \qquad (4.1.)$$

where x and y represent population size and employment rate, respectively.

Next, we assume that for a declining city the urban attractiveness has a negative growth rate, while this negative trend may be compensated by a rise in the employment potential, i.e.

$$\dot{z} = -\beta_1 z + \gamma y \qquad (4.2.)$$

where z represents the above mentioned immigration rate. The growth rate of immigration related to employment (i.e., γ) is assumed to be positively correlated with agglomeration economies emerging from city size, i.e.

$$\gamma = \beta_2 x \qquad (4.3.)$$

so that we obtain for (4.2.) the following expression:

$$\dot{z} = -\beta_1 z + \beta_2 xy \qquad (4.4.)$$

Finally, the employment potential of a declining city is assumed to have a negative growth rate, which may be reinforced by a high immigration rate from outside, but which may also be positively influenced by a rise in city size, i.e.

$$\dot{y} = -\delta_1 y - \epsilon z + \delta_3 x \qquad (4.5.)$$

Next, we assume that the (negative) growth rate of employment with respect to immigration rate (i.e., ϵ) is affected by synergetic effects related to city size, i.e.,

$$\epsilon = \delta_2 x \qquad (4.6.)$$

so that at the end we obtain the following expression for (4.5.):

$$\dot{y} = -\delta_1 y - \delta_2 xz + \delta_3 x \qquad (4.7.)$$

Now it can easily be seen that (4.1.), (4.4.) and (4.7.) are essentially Lorenz equations. If $\sigma_1 = \sigma_2 = \sigma$, $\delta_1 = \delta_2 = 1$, $\delta_3 = r$, $\beta_1 = b$ and $\beta_2 = 1$, we find exactly the Lorenz model described in subsection 2.4.

Now it may be interesting to investigate the stability conditions and steady state solutions of our model. Two directions can be followed here, viz. an analysis of the dynamic properties of our model in order to identify the conditions for and the values of steady state solutions (see Annex A for a formal derivation) or a series of simulation experiments in order to study the long-term behaviour of our model for varying parameter values (see section 5 for some numerical results).

The foregoing model of urban decline can also be used in an alternative way, viz. by trying to incorporate our model in an optimal control framework (see section 6). Up till now, the latter research direction has not yet been undertaken.

5. Results of a Simulation Experiment

In general, due to lack of data, it will be difficult to provide an econometric estimation of dynamic models for urban evolution, so that resort has to be taken to simulation analysis. In this section some results from various simulation experiments will briefly be presented. Two simulation runs will be described.

(a) Modest urban decline

For this simulation the following parameter values will be assumed:

$\sigma_1 = 0.1$	$\delta_1 = 0.1$	$\beta_1 = 0.01$
$\sigma_2 = 0.001$	$\delta_2 = 0.005$	$\beta_2 = 0.0001$
	$\delta_3 = 0.001$	

The initial values are:	$x = 100$
	$y = 0.5$
	$z = 0.1$

The results are printed in Figure 6.

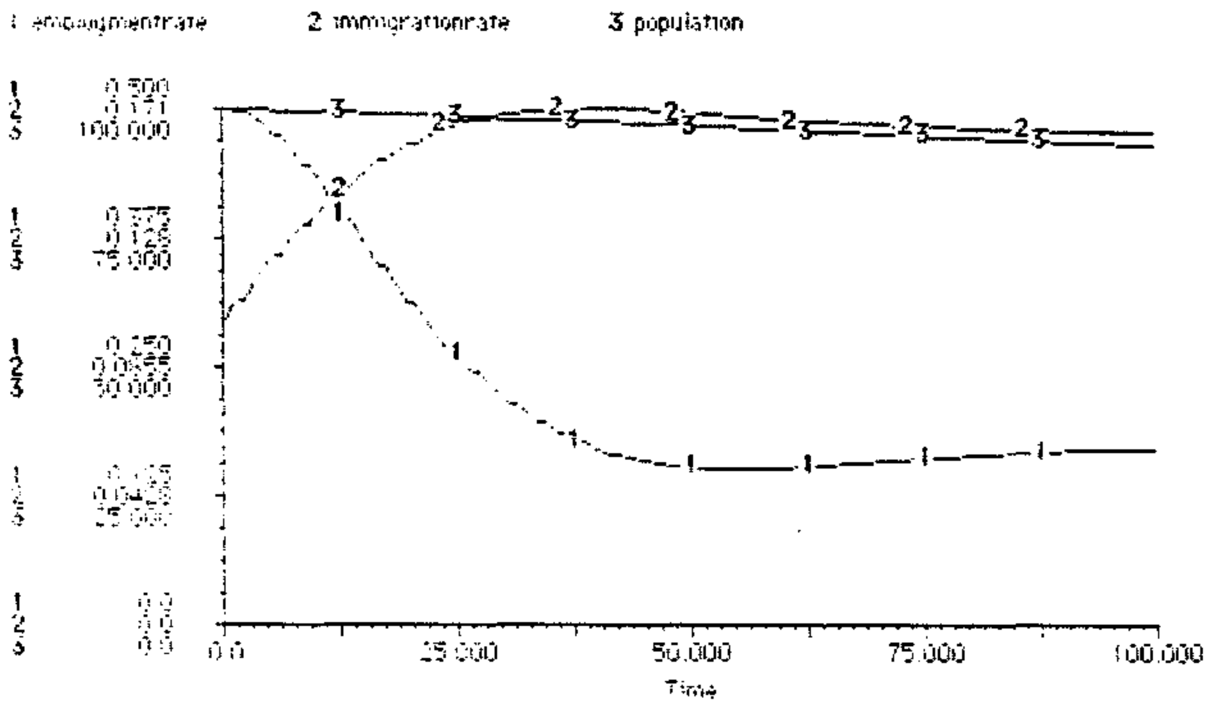

Figure 6. Modest urban decline

Figure 6 shows that - given the parameter values and the initial values given above - the urban system concerned shows a gradual decline for the total population, a significantly decreasing pattern for the unemployment rate and a clear growth pattern for the immigration rate (followed by a slight decline in later periods). The net result for the urban system appears to be one of a modest decline.

(b) <u>Chaotic urban decline</u>

Here we assume the following parameter values:

$\sigma_1 = 0.1$	$\delta_1 = 0.1$	$\beta_1 = 0.1$
$\sigma_2 = 0.01$	$\delta_2 = 0.01$	$\beta_2 = 0.001$
	$\delta_3 = 0.01$	

The initial values of the variables are again the same. The results are plotted in Figure 7.

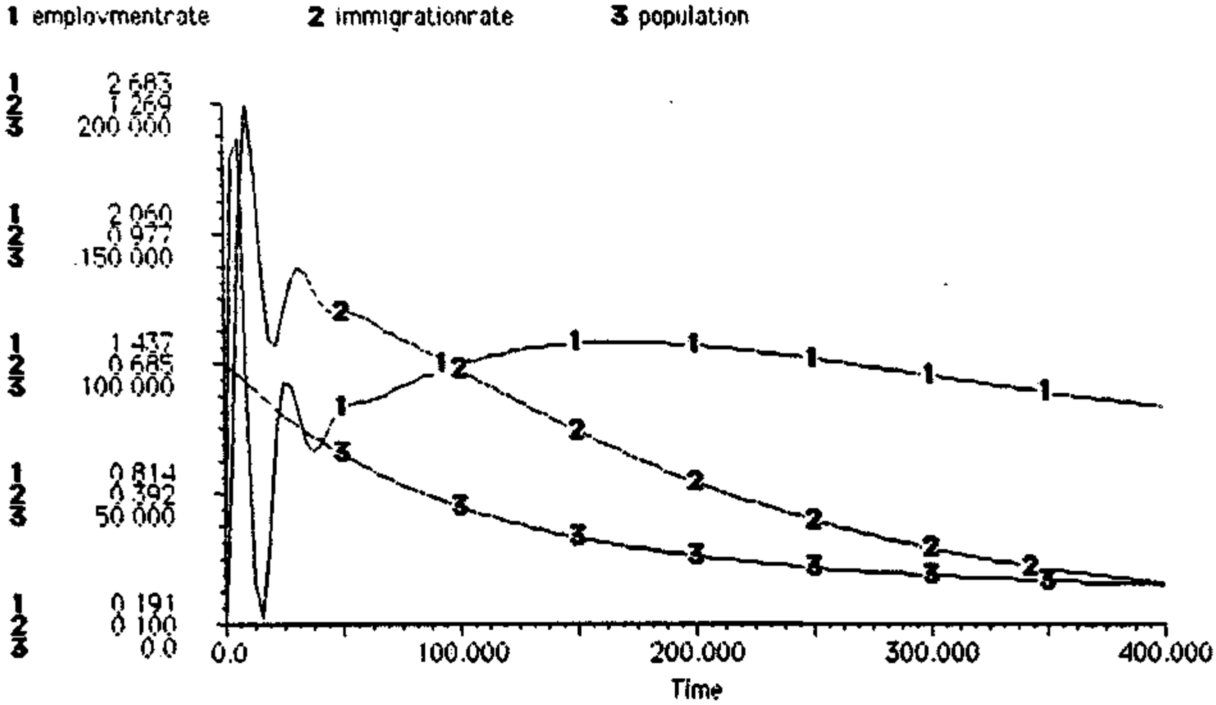

Figure 7. Chaotic urban decline

These results show a significant steady decrease of urban population. However, the immigration rates and employment rates show - in initial periods - contrasting chaotic behaviour, while in later periods declining immigration and employment rates lead to a structural decline of the city.

These results make clear that chaotic behaviour is not a necessity, but may emerge as a result of specific critical parameter values and initial conditions. The question whether the evolution of such an urban system can be controlled in a more smooth way will be discussed in the next section in the framework of optimal control theory.

6. An Optimal Control Formulation of a 'Chaos' Problem

In this section we will examine a general dynamic system in which the state variables behave according to a (general) Lorenz model. As we have seen before, this situation could be a particular case of urban decline as discussed in section 4. We assume that we can control this system by maximising a general utility function incorporating for the sake of simplicity - in addition to the three state variables - only the three parameters of the original Lorenz system (see (2.4)) as control variables. For the sake of presentation we will use a well-behaved logarithmic function, which is very often used in the utility literature (see also Somermeijer and Banninck, 1973).

Therefore we have the following optimal control problem:

$$\text{Max } U = \int_0^T (\alpha_1 \ \ell n \ r + \alpha_2 \ell n \ y + \eta_1 \ell n \ \sigma + \eta_2 \ell n \ x + \kappa_1 \ell n \ b + \kappa_2 \ell n \ z) \ dt \tag{6.1}$$

$$\text{s.t.} \qquad \begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned}$$

where α_1, α_2, η_1, η_2, κ_1, κ_2 are trade-off coefficients in the utility function. The use of our generalized Lorenz model (including more coefficients) from section 4 would not lead to any additional computational problems in this optimal control model.

The Hamiltonian H associated with (6.1) is:

$$\begin{aligned} H = \alpha_1 \ \ell n \ r + \alpha_2 \ \ell n \ y + \eta_1 \ \ell n \ \sigma + \eta_2 \ \ell n \ x + \kappa_1 \ \ell n \ b + \kappa_2 \ell n \ z \\ + \lambda \ (-\sigma x + \sigma y) + \mu \ (-xz + rx - y) + \psi \ (xy - bz) \end{aligned} \tag{6.2}$$

where λ, μ, ψ are the costate variables associated with $\dot{x}$, $\dot{y}$, $\dot{z}$ respectively.

Then the first-order (necessary) conditions for optimality are:

$$\frac{\partial H}{\partial \sigma} = \frac{\partial H}{\partial r} = \frac{\partial H}{\partial b} = 0 \tag{6.3}$$

so that we have:

$$\left.\begin{aligned} \frac{\partial H}{\partial \sigma} &= \frac{\eta_1}{\sigma} - \lambda x + \lambda y = 0 \\ \frac{\partial H}{\partial r} &= \frac{\alpha_1}{r} + \mu x = 0 \\ \frac{\partial H}{\partial b} &= \frac{\kappa_1}{b} - z\psi = 0 \end{aligned}\right\} \tag{6.4}$$

Thus we can now easily obtain the following expression for the optimal values of the control variables:

$$\sigma = \frac{\eta_1}{\lambda(x-y)}$$

$$r = \frac{-\alpha_1}{\mu x} \tag{6.5}$$

$$b = \frac{\kappa_1}{z\psi}$$

The transversality conditions are then the following:

$$\dot{\lambda} = - \frac{\partial H}{\partial x} = - (\frac{\eta_2}{x} - \lambda\sigma - z\mu + r\mu + \psi y)$$

$$\dot{\mu} = - \frac{\partial H}{\partial y} = - (\frac{\alpha_2}{y} + \lambda\sigma - \mu + \psi x) \tag{6.6}$$

$$\dot{\psi} = - \frac{\partial H}{\partial z} = - (\frac{\kappa_2}{z} - x\mu - b\psi)$$

If we now substitute the optimal values (6.5) into (6.6) and into the equations for the state variables from (6.1), we get the following six-dimensional differential system:

$$\dot{x} = - \frac{\eta_1}{\lambda}$$

$$\dot{y} = -xz - y - \frac{\alpha_1}{\mu}$$

$$\dot{z} = xy - \frac{\kappa_1}{\psi}$$

$$\dot{\lambda} = \frac{\alpha_1 - \eta_2}{x} + \frac{\eta_1}{x-y} - \psi y + z\mu \tag{6.7}$$

$$\dot{\mu} = -\psi x - \frac{\eta_1}{x-y} - \frac{\alpha_2}{y} + \mu$$

$$\dot{\psi} = \mu x + \frac{\kappa_1 - \kappa_2}{z}$$

The linearized system - on the basis of a first-order Taylor expansion - around a possible steady state x^*, y^*, z^*, λ^*, μ^*, ψ^* appears to be equal to:

$$
\begin{bmatrix} x \\ y \\ z \\ \cdots \\ \lambda \\ \mu \\ \psi \end{bmatrix}^{\bullet}
=
\left[\begin{array}{ccc:ccc}
0 & 0 & 0 & \frac{\eta_1}{\lambda^{*2}} & 0 & 0 \\
-z^* & -1 & -x^* & 0 & \frac{\alpha_1}{\mu^{*2}} & 0 \\
y^* & x^* & 0 & 0 & 0 & \frac{\kappa_1}{\psi^{*2}} \\
\hdashline
\left(\frac{\eta_2-\alpha_1}{x^{*2}} - \frac{\eta_1}{(x^*-y^*)^2}\right) & \left(-\psi^* + \frac{\eta_1}{(x^*-y^*)^2}\right) & \mu^* & 0 & z^* & -y^* \\
\left(-\psi^* + \frac{\eta_1}{(x^*-y^*)^2}\right) & \left(\frac{\alpha_2}{y^{*2}} - \frac{\eta_1}{(x^*-y^*)^2}\right) & 0 & 0 & 1 & -x^* \\
\mu^* & 0 & \frac{\kappa_2-\kappa_1}{z^{*2}} & 0 & x^* & 0
\end{array}\right]
\begin{bmatrix} x - x^* \\ y - y^* \\ z - z^* \\ \cdots \\ \lambda - \lambda^* \\ \mu - \mu^* \\ \psi - \psi^* \end{bmatrix}
\qquad (6.8)
$$

The properties of system (6.8) deserve a closer examination. It is evident that in matrix (6.8) the trace is equal to zero. When the trace of a system equals zero for all parameter values it is plausible that a case of center dynamics may exist. This certainly happens in the case of a system of two differential equations (see Guckenheimer and Holmes, 1983 and Kaplan, 1958), provided the determinant of the linearized transition matrix (i.e., the Jacobian) is positive (i.e., a situation of complex roots).

In our specific case of more than two dimensions, the situation is much more complicated and a straightforward conclusion cannot be inferred. The results depend on the pre-specified values of the parameters. Consequently both complex and real roots may emerge, but in case of complex roots and of a zero trace we may face center dynamics and hence oscillating behaviour (viz. neutral stability).

7. Concluding Remarks

At the end of this paper some reflective remarks are in order. First, models based on the theory of chaos do not ensure the existence of chaos, but at best the potential emergence of unexpected dynamic behaviour, depending on initial conditions and on critical parameter values. Especially the extreme sensitivity on incremental changes is noteworthy, which puts a high burden on specification analysis for models describing economic and social systems in a behavioural context. The predictive value of such chaotic models may sometimes be questioned, especially since - despite their deterministic nature - they may lead to unanticipated results. An important lesson drawn from the foregoing is that major attention has to be given to specification analysis in the social sciences (see also Blommestein, 1986).

A second point concerns the relevance of chaotic models. The models reviewed in sections 2 and 3 were relatively simple in nature and contained only a few equations. In general, however, realistic models for economic and social phenomena are much richer in scope. Thus here we face the intriguing question of overall stability of a system's model, if one of the subsystems is described by a chaotic model (see de Wolff, 1984). Would the overall steady state of a comprehensive system's model be endangered by potential chaotic behaviour of a small 'niche' in the system or, inversely, would the potential chaotic behaviour of a small subsystem's model be reduced by a stable 'environment'? Such research questions are extremely relevant in economics, as global instability of an economic system is not a likely feature, but local perturbances in specific subsectors of the economy are much more plausible. Seen from this perspective, theory of chaos opens a wide spectrum for future economic research on dynamics systems behaviour.

Annex A. Steady State of Solutions for a Generalized Lorenz Systems

In this Annex we will investigate the existence of fixed point solutions for our system of generalized Lorenz equations.

It is well known that the Lorenz equations presented before possess the obvious basic steady state solution $x = y = z = 0$. With this solution the onset of convection is given for $r=1$.

When $r>1$, the Lorenz equations in (2.4) possess two additional steady states, viz:

$$\left.\begin{aligned} x &= y = \pm\sqrt{b(r-1)} \\ z &= r-1 \quad , \end{aligned}\right\} \qquad (A.1)$$

around which - for particular values of the parameters - two spirals emerge (see also Figure 3).

We will now explore in an analogous way our dynamic system:

$$\left.\begin{aligned} \dot{x} &= -\sigma_2 x + \sigma_1 y \\ \dot{y} &= -\delta_2 xz + \delta_3 x - \delta_1 y \\ \dot{z} &= \beta_2\, xy - \beta_1 z \end{aligned}\right\} \qquad (A.2)$$

It is clear that also here the obvious steady state solution $x=y=z=0$ exists. The linear transition matrix of (A.2), based on a Taylor series around a steady state (x_o, y_o, z_o), is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^{\bullet} = \begin{bmatrix} -\sigma_2 & \sigma_1 & 0 \\ \delta_3 - \delta_2 z_o & -\delta_1 & -\delta_2 x_o \\ \beta_2 y_o & \beta_2 x_o & -\beta_1 \end{bmatrix} \begin{bmatrix} x-x_o \\ y-y_o \\ z-z_o \end{bmatrix} \qquad (A.3)$$

Consequently the characteristic equation of (A.3) for the solution $x=y=z=0$ is:

$$[\lambda + \beta_1]\ [\lambda^2 + (\sigma_2+\delta_1)\ \lambda + \sigma_2\delta_1 - \sigma_1\delta_3] = 0 \qquad (A.4)$$

It can easily be seen that the equation has three real roots when $\sigma_1\delta_3>0$. This system is stable, if $\sigma_1\delta_3<\sigma_2\delta_1$. When $\sigma_2\delta_1=\sigma_1\delta_3$, we have a critical value after which motion begins (for $\sigma_1\delta_3>\sigma_2\delta_1$). Furthermore when $\sigma_1\delta_3>\sigma_2\delta_1$, system (A.2) possesses two additional steady state solutions, viz.

$$\left.\begin{aligned} x &= \pm\sqrt{\frac{\beta_1}{\beta_2}\ \frac{\sigma_1\delta_3-\sigma_2\delta_1}{\delta_2\ \sigma_2}} = \frac{\sigma_1}{\sigma_2}\ y \\ z &= \frac{\sigma_1\delta_3-\sigma_2\delta_1}{\sigma_1\ \delta_2} \\ y &= \frac{\sigma_2}{\sigma_1}\ x \end{aligned}\right\} \qquad (A.5)$$

The previous solutions may be tested on their stability conditions by means of simulation experiments (see also section 6).

Next we show that the behaviour of the solution from (A.5) is the same as in the original Lorenz model. In that case, a linearization of the positive vector field at the fixed

point $= +\sqrt{\frac{\beta_1}{\beta_2}\ \frac{\sigma_1\delta_3-\sigma_2\delta_1}{\delta_2\sigma_2}} = \frac{\sigma_1}{\sigma_2}\ y,\qquad z = \frac{\sigma_1\delta_3-\sigma_2\delta_1}{\sigma_1\delta_2}$,

is:

$$M = \begin{bmatrix} -\sigma_2 & \sigma_1 & 0 \\ \frac{\sigma_2}{\delta_1}\delta_1 & \delta_1 & -\delta_2\sqrt{\frac{\beta_1}{\beta_2}\ \frac{\sigma_1\delta_3-\sigma_2\delta_1}{\delta_2\sigma_2}} \\ \beta_2\ \frac{\sigma_2}{\sigma_1}\sqrt{\frac{\beta_1}{\beta_2}\cdot\frac{\sigma_1\delta_3-\sigma_2\delta_1}{\delta_2\sigma_2}} & \beta_2\sqrt{\frac{\beta_1}{\beta_2}\ \frac{\sigma_1\delta_3-\sigma_2\delta_1}{\delta_2\sigma_2}} & -\beta_1 \end{bmatrix}$$

(A.6)

The characteristic polynomial of the matrix (A.6) is:

$$\lambda^3 + (\sigma_2+\delta_1+\beta_1)\lambda^2 + (\frac{\beta_1\sigma_2{}^2 + \beta_1\sigma_1\delta_3}{\sigma_2})\ \lambda + 2\beta_1\ (\sigma_1\delta_3-\sigma_2\delta_1) = 0, \tag{A.7}$$

which has one negative and two complex roots.

It is straightforward to see, by applying the Hopf bifurcation theorem (see Marsden and McCracken, 1976), that for $\sigma_2>\beta_1+\sigma_1$, a Hopf bifurcation occurs at the following point:

$$\delta_3^* = \frac{\sigma_2^2\ (\sigma_2+\beta_1+3\delta_1)}{\sigma_1\ (\sigma_2-\beta_1-\delta_1)} \tag{A.8}$$

Acknowledgement:
The authors are indebted to Professor Hans-Walter Lorenz (Göttingen) who provided some interesting comments on previous ideas for this paper. The computer assistance offered by Ron Janssen (Amsterdam) is also gratefully acknowledged. Finally, the grant provided by C.N.R. no. 87.01286.11 has to be mentioned here.

References

Batty, N., and P.A. Longley, The Fractal Simulation of Urban Structure, Environment and Planning A, vol. 18, 1986, pp. 1143-1179.

Benhabib, J., and R.H. Day, Erratic Accumulation, Economics Letters, vol. 6, 1980, pp. 113-117.

Benhabib, J., and R.H. Day, A Characterization of Erratic Dynamics in the Overlapping Generations Model, Journal of Economic Dynamics and Control, vol. 4, 1982, pp. 37-55.

Blommestein, H.J., Eliminative Econometrics, Ph.D. diss., Dept. of Economics, Free University, Amsterdam, 1986.

Crutchfield, J.P., Farmer, J.D., Packard N.H., and R.S. Shaw, Chaos, Scientific American, vol. 255, no. 6, 1986, pp. 46-57.

Cugno, F., and L. Montrucchio, Cyclic and Chaotic Behaviour in Price and Quantity Adjustment Models, Ricerche Economiche, vol. 38, 4, 1984, pp. 553-564.

Cvitanovic, P. (Ed.), Universality in Chaos, Adam Hilger, Bristol, 1984.

Day, R.H., Irregular Growth Cycles, American Economic Review, vol. 72, 1982, pp. 406-414.

Dendrinos, D.S., Turbolence and Fundamental Urban/Regional Dynamics, Paper presented at the American Association of Geographers, Washington DC, April 1984.

Dendrinos, D.S., On the Incongruous Spatial Employment Dynamics, Technological Change, Employment and Spatial Dynamics, (P. Nijkamp ed.), Springer-Verlag, Berlin, 1986, pp. 321-339.

Dendrinos, D.S., and M. Sonis, The Onset of Turbolence in Discrete Relative Multiple Spatial Dynamics, Applied Mathematics and Computation, vol. 22, 1987, pp. 25-44.

Dendrinos, D.S., and M. Sonis, Non Linear Discrete Relative Population Dynamics of the V.S. Regions, Journal of Applied Mathematics and Computation, 1988 (forthcoming).

Domanski, R., and A.P. Wierzbicki, Self-Organization in Dynamic Settlement Systems, Papers of the Regional Science Association, vol. 51, 1983, pp. 141-160.

Eckmann, J.P., and D. Ruelle, Ergodic Theory of Chaos and Strange Attractors, Review of Modern Physics, vol. 57, no. 3, 1985, pp. 617-656.

Gilpin, M.E., Spiral Chaos in a Predator-Prey Model, American Nature, 113-1979, pp. 306-308.

Goodwin, R.M., Krüger M., and A. Vercelli (Eds.), Non Linear Models of Fluctuating Growth, Springer-Verlag, Berlin, 1984.

Grandmont, J.M., Periodic and Aperiodic Behaviour in Discrete One Dimensional Dynamic Systems, Working Paper CEPREMAP, no. 8317, Paris, 1984.

Guckenheimer J. and P. Holmes, Non-Linear Oscillations, Dynamical System and Bifurcation of Vector Fields, Springer Verlag, Berlin, 1983.

Guckenheimer, J., Oster, G., and A. Ipatchi, The Dynamics of the Density Dependent Population Models, Journal of Mathematical Biology, vol. 4, 1977, pp. 101-147.

Hao, B.L., (Ed.), Chaos, World Scientific Publishing Co., Singapore, 1984.

Hénon, M., A Two-dimensional Mapping with a Strange Attractor, Communications in Mathematical Physics, vol. 50, 1976, pp. 69-77.

Hénon, M., and Y. Pomeau, Two Strange Attractors with a Simple Structure, Turbolence and Navier Stokes Equations (R. Temam, ed.), Springer-Verlag, Berlin, 1975, pp. 30-68.

Holden, A.V., (Ed.), Chaos, Manchester University Press, Manchester, 1986.

Holden, A.V., and M.A. Muhamad, A Graphical Zoo of Strange and Peculiar Attractors, Chaos (A.V. Holden, ed.), Manchester University Press, Manchester, 1986, pp. 15-35.

Kaplan, W., Ordinary Differential Equations, Addison-Wesley, Reading, Mass, 1958.

Li, T.Y., and J.A. Yorke, Period Three Implies Chaos, American Mathematical Monthly, vol. 82, no. 10, 1985, pp. 985-992.

Lichtenberg, A.J., and M.A. Lieberman, Regular and Stochastic Motion, Springer Verlag, Berlin, 1983.

Lorenz, E.N., Deterministic Nonperiodic Flow, Journal of the Atmospheric Sciences, vol. 20, no. 2, 1963, pp. 130-141.

Lorenz, H.W., Strange Attractors in a Multisector Business Cycle Model, Paper presented at the Workshop "Advances in the Analysis of Economic Dynamic Systems", Venice, 1986.

Lung, Y., Complexity and Spatial Dynamics Modelling, Annals of Regional Science, 1988 (forthcoming).

Mandelbrot, B., The Fractal Geometry of Nature, W.H. Freeman and Company, New-York, 1977.

Marsden, J.E., and M. McCracken, The Hopf Bifurcation and its Applications, Springer Verlag, Berlin, 1976.

May, R., Simple Mathematical Models with Very Complicated Dynamics, Nijkamp, P., Long-Term Economic Fluctuations: A Spatial View, Socio-Economic Planning Sciences, vol. 21, no. 3, 1987, pp. 189-197.

Nijkamp, P., J. Poot, and J. Rouwendal, R&D Policy in Space and Time, A Non-Linear Dynamic Spatial Model, Research Memorandum, Dept. of Economics, Free University, 1988.

Ott, E., Strange Attractors and Chaotic Motions of Dynamical Systems, Review of Modern Physics, vol. 53, no. 4, 1981, pp. 655-671.

Pacini, P.M., Recent Approaches in Dynamical System Theory in Economics with Emphasis on Chaos: A Critic Review, European University Institute, Florence, 1986 (mimeographed).

Peitgen, H.O., and P.H. Richter, The Beauty of Fractals, Springer-Verlag, Berlin, 1986.

Rössler, O.E., An Equation for Continuous Chaos, Physics Letters, 57A, 1976, 397.

Ruelle, D., and F. Takens, On the Nature of Turbolence, Communications in Mathematical Physics, vol. 20, 1971, pp. 167-192.

Schaffer, W.M, and M. Kot, Differential Systems in Ecology and Epidemiology, Chaos (A.V. Holden, ed.), Manchester University Press, Manchester, 1986, pp. 158-178.

Somermeijer, W.H., and R. Bannink, A Consumption-Savings Model and its Applications, North-Holland Publ. Co., Amsterdam, 1973.

Sonis, M., and D.S. Dendrinos, Period-Doubling in Discrete Relative Spatial Dynamics and the Feigenbaum Sequence, Mathematical Modelling: an International Journal, 1987 (Sept.) (forthcoming).

Sparrow, C.T., The Lorenz Equations: Bifurcations, Chaos and, Strange Attractors, Springer, New York, 1982.

Stutzer, M., Chaotic Dynamics and Bifurcation in a Macro Model, Journal of Economic Dynamics and Control, vol. 2, 1980, pp. 253-275.

Weiss, N.O., Dynamics of Convection, Proceedings Royal Soc. Lond. A, no. 413, 1987, pp. 71-85.

White, R.W., Transitions to Chaos with Increasing System Complexity: The Case of Regional Industrial Systems, Environment and Planning A, vol. 17, 1985, pp. 387-396.

Wollf, P. de, Niet-lineaire vergelijkingen, Toegepaste Economie (H. den Hartog en J. Weitenberg, eds.) Central Planning Bureau, The Hague, 1984, pp 345 - 361

Yorke, J.A., and E.D. Yorke, Chaotic Behaviour and Fluid Dynamics, Hydrodynamics Instabilities and the Transition to Turbolence (Swinney H.L., and J.P. Gollub, eds), Springer-Verlag, Berlin, 1981, p. 77-91.